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Efficient evaluation of diffuse view factors for radiation

V. RAMMOHAN RAO† and V. M. K. SASTRI‡

Department of Mechanical Engineering, Indian Institute of Technology, Madras 600 036, India

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Abstract—This paper presents a numerical method of evaluating view factors between planar surfaces which is computationally efficient and quite general to program on a computer. The method, which is based on Gaussian quadrature to perform the contour integration is extended to surfaces with curved boundaries. The performance of various quadrature formulas viz. trapezoidal, Simpson and Gaussian, have been compared for performance by applying them to sample problems. The Gaussian quadrature method with nonlinear transformation to map the boundary has been found to be the most accurate, computationally faster and very general. As an application, the shape factor between two elliptic surfaces has been evaluated.

INTRODUCTION

In radiation analysis of many engineering problems, diffuse approximation is common. In such situations the view factor, which is the fraction of diffuse radiation leaving the surface and reaching the interacting surface, plays an important role. The solution for radiative transfer coupled with other modes of heat transfer in enclosures is normally iterative due to the nonlinear nature. Usage of inaccurate view factors magnifies the errors in the final solution [1]. The view factors for many simple geometries have been calculated analytically and tabulated in standard texts, e.g. [2, 3]. Accurate determination of view factors by numerical means has been a topic of research, since analytical solution is not possible for many of the geometries of practical interest.

The importance of numerical evaluation of view factors is apparent from earlier papers, e.g. [6–11]. Chung and Kim [6] applied the finite element method to evaluate view factors. As was pointed out by Ambirajan and Venkateshan [9], the results obtained were not convincingly accurate even with a fine mesh. In an interesting paper by Shapiro [7], accuracy and computational time were compared for the area integral method and line integral method applied to a problem of two directly opposed squares (test problem 1 in this paper). Recently, Byrd presented view factor algebra for two arbitrarily sized non-opposing parallel rectangular surfaces. The view factor between a small rectangular plane to a triangular surface perpendicular to the rectangular plane was calculated by Noboa *et al.* [10] using shape factor algebra. This situation was similar to the one encountered in radiative analysis of attics. In this work, the common edge which poses singularity was conveniently avoided. The most important observation which is made from these works is that there still does not exist a general numerical method of computing view factors accurately for any geometry. In a more recent paper by Ambirajan and Venkateshan [9], they presented a more general method which uses a Romberg integration formula based on the trapezoidal rule. An analytical formula was presented for the situation where the surfaces have a common edge. A method of treating curved surfaces was also presented. After a detailed survey of previous work, it is felt that it is possible to come up with a general method of diffuse view factor evaluation which will be computationally efficient and which can be applied to any geometry (at least for plane surfaces) with desired accuracy.

CONTOUR INTEGRATION METHOD

It has been shown by Sparrow [5] that the view factor between two surfaces can be given by the following formula, which has been popularly referred to as the contour integration formula in text books on radiation.

$$F_{1-2} = \frac{1}{2\pi A_1} \oint_{\Gamma_1} \oint_{\Gamma_2} \ln s \, \mathbf{dr}_1 \cdot \mathbf{dr}_2. \tag{1}$$

In this, the subscripts 1 and 2 refer to the two surfaces under consideration, s is the distance between two line elements on each contour and $\mathbf{dr}$ is the elemental length vector. This equation is the transformed form of the definition of the view factor between surfaces 1 and 2 using Stoke's theorem which is given by

† Present address: Department of Mechanical Engineering, Auburn University, Auburn, AL 36849, U.S.A.

‡ Alexander von Humboldt Fellow, ITW 0507, Universitat Stuttgart, 70569 Stuttgart, Germany. Author to whom correspondence should be addressed.

NOMENCLATURE

A	area of the surface
A and B	semi major and minor axes of the ellipses in the application problem.
a_{x1}, b_{x1}, c_{x1}	constants in the transformation for surface 1
F	diffuse view factor
L	length of each side in test problems 1 and 2
L_c	length of the common edge in test problem 2
N	number of elements on each contour defined in the text.
R	radius of the circle in test problem 3
$\mathbf{r}$	length vector
s	distance between two points on each contour
x, y, z	Cartesian coordinates of a point on the surface/contour.

Greek symbols

Γ	contour
β	angle between the normal to the surface and the line connecting two elemental surfaces
χ, η, ζ	transformed coordinates of a point on the surface/contour.

Subscript

1, 2	refer to surfaces 1 and 2, respectively.

$$F_{1-2} = \frac{1}{A_1}\int_{A_1}\int_{A_2}\frac{\cos\beta_1\cos\beta_2\,\mathrm{d}A_1\,\mathrm{d}A_2}{\pi s^2}. \quad (2)$$

The accuracy of the view factor evaluated using the above formulas depends on the efficiency of the numerical scheme used. The obvious advantage of transforming equation (2) to equation (1) is that quadruple integration has become a double contour integration. An extensive comparison of computational times needed to evaluate the area integral in equation (2) and the line integral in equation (1) has been made by Shapiro [7] who concluded that the area integral method needs orders of magnitude of more computation time than the line integral method. There is another major advantage in doing so, which will be discussed later in the paper.

Numerical evaluation of the contour integral in equation (1) consists of dividing the contours of the two surfaces into a finite number of line elements. The contribution of all the elements on both the contours to the integral is added up to get the view factor between the two surfaces. A closer observation of equation (1) reveals that the accuracy in computing the view factor depends on:

(1) How closely one can approximate the variation of $\ln(s)$ within the elemental intervals. It should be noted that the function $\ln(s)$ varies very seriously when the contours are very close and becomes $-\infty$ where two contours touch each other.

(2) How closely one can follow the contours. For surfaces with straight contours this does not impose any error as the traditional numerical integration techniques follow the straight contours exactly.

With reference to the above points, various numerical integration methods are compared in the following.

QUADRATURE FORMULAS

Consider two line elements on the contours of each surface as shown in Fig. 1. Three points are shown on

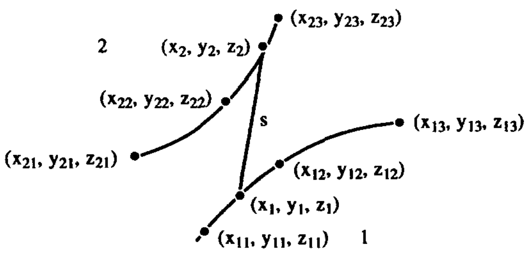

Fig. 1. Sketch showing three points on each contour for contour integration formula.

each of the elements. For this situation, equation (1) can be expanded as

$$2\pi A_1 \Delta F_{1-2} = \oint_{x11}^{x13}\oint_{x21}^{x23} \ln(s)\,\mathrm{d}x_1\,\mathrm{d}x_2 + \oint_{y11}^{y13}\oint_{y21}^{y23} \times \ln(s)\,\mathrm{d}y_1\,\mathrm{d}y_2 + \oint_{z11}^{z13}\oint_{z21}^{z23} \ln(s)\,\mathrm{d}z_1\,\mathrm{d}z_2 \quad (3)$$

where $s = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$ and ΔF_{1-2} is the line integral contribution from the two elements considered. The view factor F_{1-2} will be the sum of such contributions from each element on surface 1 to each element on the surface 2.

Three methods are considered in this paper to evaluate the integrals in equation (3), viz. (1) trapezoidal, (2) Simpson and (3) Gaussian quadrature. These are standard techniques of numerical quadrature and, hence, are only salient features which are relevant to the present problem are highlighted here.

Trapezoidal method (TZM)

Trapezoidal method is the simplest of all the quadrature formulas and needs two points on each contour element. This involves linear approximation of the function, i.e. $\ln(s)$ here within the limits. For a single integration it needs evaluation of the function to be carried out twice and hence four times for a double integration. This method has been used in ref. [9] to perform Romberg integration.

Simpson's method (SIM)

The function to be integrated is approximated by a quadratic function through three points within the limits of integration. Each integration requires evaluation of function nine times.

Gaussian quadrature

Gaussian quadrature is the most accurate method for a given number of points on the elements. An n-point quadrature formula approximates the function by $(2n-1)$th degree polynomial. A one-point formula (GAUSS1) evaluates the integral as accurate as the TZM with just one point on each element. A two-point formula (GAUSS2) approximates the function with a polynomial of degree three and a three-point formula (GAUSS3) with degree five. To get the same accuracy, the SIM needs four and six points, respectively. In general, for a desired accuracy, SIM needs four times more computation time compared to the Gaussian method.

The above comparison is true for surfaces with straight contours. As mentioned earlier, additional error is introduced due to the approximation of the contour as a straight line between the points. Very large computation times are required with the above methods if the contours are curved.

Another complication is when the two surfaces share a common edge. In this case, the distance between any two points on each contour is zero and hence numerical integration becomes impossible. However, there exists an exact integration formula as shown in ref. [9] which is given by

$$2\pi A_1 \Delta F_{1-2} = L_c^2(1.5 - \ln L_c). \tag{4}$$

This is the contribution to the overall view factor from the element on the common edge of the two surfaces. All that it needs is the length of the common edge.

Gaussian quadrature with nonlinear transformation

The Gaussian quadrature needs the integration limits to be between -1 and $+1$ and hence demands a transformation of global coordinates to local ones. One obvious way is a linear transformation which has been used presently for GAUSS1, GAUSS2 and GAUSS3. In this, the Gaussian points do not exactly lie on the contour, if the contours are curved. An alternative to make the integration follow the contour more accurately is by use of higher order transformation. This is similar to the use of isoparametric elements in the finite element method. In the present study, the results are reported for only a quadratic transformation. Though in principle, higher order transformation improves the accuracy, it is found that the improvement is better with an increase in the number of elements than with the order of transformation beyond the quadratic. Referring to Fig. 1, the global coordinates of the points on the two contours are transformed using the following:

$$\begin{aligned} x_1 &= a_{x_1}\chi_1^2 + b_{x_1}\chi_1 + c_{x_1} \\ y_1 &= a_{y_1}\eta_1^2 + b_{y_1}\eta_1 + c_{y_1} \\ z_1 &= a_{z_1}\zeta_1^2 + b_{z_1}\zeta_1 + c_{z_1} \end{aligned} \tag{5}$$

for contour 1 and a similar one for contour 2. The constants

$$a_{x_1}, a_{x_2}, \ldots$$

are evaluated using the limits of the transformed variables which are between -1 and $+1$. The intermediate values of χ, η, etc., i.e. χ_{12}, η_{12}, etc. can be anything arbitrary between -1 and $+1$ and are selected as zero for convenience. For this transformation, three points on each contour are needed and must be located on the contours.

With the transformed variables, equation (3) can be recast as

$$\begin{aligned} 2\pi A_1 \Delta F_{1-2} = & \int_{-1}^{+1}\int_{-1}^{+1} (2a_{x_1}\chi_1 + b_{x_1}) \ln(s)\, d\chi_1\, d\chi_2 \\ & + \int_{-1}^{+1}\int_{-1}^{+1} (2a_{y_1}\eta_1 + b_{y_1}) \ln(s)\, d\eta_1\, d\eta_2 \\ & + \int_{-1}^{+1}\int_{-1}^{+1} (2a_{z_1}\zeta_1 + b_{z_1}) \ln(s)\, d\zeta_1\, d\zeta_2. \end{aligned} \tag{6}$$

It can be observed that the function $\ln(s)$, with the same expression for s as earlier, has been given weights which depend on the location on the contours. This is again similar to the Jacobians appearing during the transformation in the finite element methods. There is an additional computational effort because of these weights. However, this is very well compensated for by not having to evaluate $\ln(s)$ for each of the three integrals. The two-point and three-point Gaussian quadrature with this transformation is referred to as GAUSS2N and GAUSS3N, respectively, in the following:

It should be noted that the motivation for this work is to evolve a procedure to compute view factors by combining the advantages of

(1) the possibility of a transformation of an area integral to a contour integral which is computationally more economical;

(2) the existence of a closed form formula for the situation of two surfaces with a common edge;

(3) the Gauss quadrature formula which is the most accurate one for the same computational effort; and

(4) a higher order transformation which will allow the integration to follow the contour accurately.

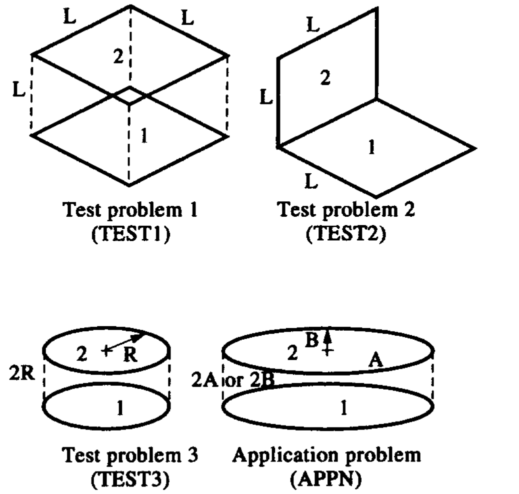

Fig. 2. Schematic of the test problems and the application problem.

RESULTS AND DISCUSSION

For the purpose of comparison of the performance of various methods mentioned earlier, three test problems are considered (refer Fig. 2), viz. (1) two parallel square plates separated by a distance equal to the side (TEST1), (2) two square plates with a common side with an included angle of 90° (TEST2), and (3) two parallel circular plates separated by a distance equal to the diameter (TEST3). Finally, the view factor between two opposing elliptic surfaces has been evaluated as an application (APPN) of the present method. All the computations are performed on a PC/AT-386 using Microsoft Fortran compiler.

Test problem 1

The first part of Table 1 shows results for TEST1. In this, N is the number of line elements on each side of the square. For this case alone, the number of elements N is varied to get a given accuracy up to eight decimal places. This is to compare the computational time required for a desired accuracy. However, the results for TZM and GAUSS1 require very large N to get the values accurate up to eight decimal places. In fact, it is found that it is not always possible to get a high accuracy using these methods, as truncation errors creep in as N increases. Computationally, this case is the simplest of all cases considered here, or earlier by other researchers [6, 7, 9]. To get the same order of accuracy SIM needs 30 elements on each side, and 10 and 5, respectively, by GAUSS2 and GAUSS3. Computational efficiencies can be seen better in the CPU time column. GAUSS3 gives a value up to nine digit accuracy within 0.7 s. The ACCURATE value is computed by taking 100 elements on each side and using GAUSS3.

Test problem 2

This case poses some challenge because of the common edge shared by the two surfaces. The integral, whether in equation (1) or (2), becomes singular on this line. For this case the value of N is taken to be 100 for all the methods. The accuracy in view factors and CPU time required can be compared from Table 1. The ACCURATE value for this case alone has been computed using four-point quadrature formula. GAUSS3 gives the best value with a computation time of two minutes. Chung and Kim [6] took a total of 1600 (40×40) area elements on each surface, used a six-point Gaussian quadrature and obtained the view factor as 0.20255 vs the accurate value of 0.200044. It was suggested that by taking a greater number of elements the desired accuracy could be achieved. When the method suggested by them was actually tried, it was found that it is not practically possible to get an accuracy of up to eight decimal places, whatever may be the number of elements and the order of the quadrature formula. Also the computational time required is of the order of hours. The reason for this is that the singularity in the area integral in equation (2) has not been treated properly. Instead, it is suggested to take a greater number of elements. The increase in the number of elements reduces the distance, s between the elements near the common edge and s^2 appears in the denominator. Other terms in the integral in equation (2) being constants, the function inside the integral varies as s^{-3} and hence, is very sensitive to s values near the singularity. Increase of the mesh size does not increase accuracy after a particular value of N and, in fact, it worsens. In the present method, the singularity is taken care of by using the exact expression, equation (4). Now the other source of error is due to very small values of distance (but not zero) s between line elements near

Table 1. Comparison of results obtained by various methods for test problems 1 and 2

	Test problem 1 (TEST1)			Test problem 2 (TEST2)		
	N	F_{1-2}	Time (s)	N	F_{1-2}	Time (s)
TZM	100	0.199821844	55.0	100	0.2037396	54.0
SIM	30	0.199824895	11.0	100	0.2002998	113.0
GAUSS1	100	0.199826424	24.0	100	0.1995222	24.0
GAUSS2	10	0.199824894	2.9	100	0.2000040	60.0
GAUSS3	5	0.199824896	0.7	100	0.2000347	120.0
ACCURATE		0.1998248957			0.2000438	

Table 2. Comparison of results obtained by various methods for test problem 3 and application problem

	Test problem 3 (TEST3)			Application problem (APPN)		
	N	F_{1-2}	Time (s)	N	F_{1-2} (APPN1)	(APPN2)
TZM	100	0.17134721	4.2	100	0.097488270	0.25413811
SIM	100	0.17138023	7.7	100	0.097501185	0.25420937
GAUSS1	100	0.17139675	2.3	100	0.097507643	0.25424502
GAUSS2	100	0.17138023	4.5	100	0.097501184	0.25420937
GAUSS3	100	0.17138023	8.1	100	0.097501185	0.25420937
GAUSS2N	100	0.17157274	1.8	100	0.097616577	0.25447263
GAUSS3N	100	0.17157275	3.2	100	0.097616576	0.25447260
ACCURATE		0.171572879			0.09761666061	0.2544728046

the common edge. The function in the line integral inequation (1) i.e. ln (s) varies as s^{-1} and hence, is less sensitive compared to that in the area integral. That means that the line integral needs a lower order quadrature formula for the same accuracy. This is where the finite element line integral method (the present one) becomes more attractive than the finite element area integral method used by Chung and Kim [6]. For example, a four-point Gaussian quadrature method for this test case needs 50 elements on each side and takes about 30 s of CPU time to give an accuracy up to seven digits. This value is at least as accurate as those reported earlier [1, 4, 9].

Test problem 3

TEST3 is considered to demonstrate the use of the present method to compute view factors accurately for curved surfaces. This case is similar to TEST1 except that the surfaces here are curved. It can be seen in the Table 2 that all methods except GAUSS2N and GAUSS3N give values which are way off from the exact one, i.e. ACCURATE [2]. The results of GAUSS2N and GAUSS3N are accurate up to seven digits. The N in the table indicates the total number of divisions of each of the circular contours. GAUSS2N and GAUSS3N yield practically the same accuracy for the same value of N, but the CPU time required is almost double for the latter. But from the previous case, i.e. TEST2, it can be concluded that GAUSS3N would have been more advantageous had the circular surfaces been touching at a point and inclined at an angle. The accuracy can be increased, either by increasing the number of elements on each surface or by taking a higher order transformation, i.e. higher order isoparametric line elements. The methods which use only linear elements (TZM to GAUSS3) need an exceptionally large number of elements to yield results accurate up to six decimal places. Ambirajan and Venkateshan [9] demonstrated a method of treating the curved geometries. The discretization method suggested is apparently laborious and needs a solution of non-linear algebraic equations which is iterative. The present method is more general, more logical and computationally efficient.

Application problem

Results are presented for two cases in Table 2. Column APPN1 is for the case when the two elliptic surfaces are kept apart by a distance equal to their major axis, and APPN2 is the case when separated by a distance equal to the minor axis. Again the number of elements on each contour is taken to be 100. The conclusions drawn are similar to test problem 3, as far as the performance of the methods is concerned. The ACCURATE values are obtained using GAUSS2N with 400 elements on each contour.

CONCLUSIONS

An efficient computational method of evaluating diffuse view factors between plane surfaces has been presented and the performance is compared for three test problems. Test problem 2 is considered to show the power of the present method to take care of the singularity due to common edges. Test problem 3 is considered to demonstrate the usefulness of the method when the surfaces are curved. As an application, the view factors for two opposing elliptic surfaces have been obtained for when the surfaces are separated by a distance equal to the major and minor axes. Based on the present work, the following conclusions can be made:

(1) The present method, which can be viewed as a finite element line integral method, is capable of computing view factors for plane surfaces which are located arbitrarily, straight or curved and share common edges practically with any desired accuracy. The method is computationally very efficient.

(2) Use of higher order quadrature is recommended when the surfaces share a common edge or if the surfaces are touching anywhere.

(3) Use of higher order transformation is recommended when the surfaces have curved contours. This is similar to the use of isoparametric elements in finite element method.

(4) It is found that the computational effort will be less when the number of elements is increased by quadratic transformation than when the order of

transformation increased. This is because a quadratic transformation fits a parabola through three points and a cubic one fits cubic parabola through four points on the contour, which essentially demands roughly the same total number of points on the contours.

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